Fully-Explicit and Self-Consistent Algebraic Reynolds Stress Model

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Abstract

A fully-explicit, self-consistent algebraic expression for the Reynolds stress, which is the exact solution to the Reynolds stress transport equation in the 'weak equilibrium' limit for two-dimensional mean flows for all linear and some quasi-linear pressure-strain models, is derived. Current explicit algebraic Reynolds stress models derived by employing the 'weak equilibrium' assumption treat the production-to-dissipation (P/ε) ratio implicitly, resulting in an effective viscosity that can be singular away from the equilibrium limit. In the present paper, the set of simultaneous algebraic Reynolds stress equations are solved in the full non-linear form and the eddy viscosity is found to be non-singular. Preliminary tests indicate that the model performs adequately, even for three dimensional mean flow cases. Due to the explicit and non-singular nature of the effective viscosity, this model should mitigate many of the difficulties encountered in computing complex turbulent flows with the algebraic Reynolds stress models.

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1 Introduction

Since its advent, the algebraic Reynolds stress approach introduced by Rodi [1] has been viewed as one of the most sophisticated closure strategy at the two-equation level of turbulence modeling. In this approach, the structural form of the Reynolds stress ($\overline{u_i u_j}$) is taken to be self-similar in space and time. That is, the anisotropy of the Reynolds stress (b_{ij}) defined as

$$b_{ij} = \frac{\overline{u_i u_j}}{2K} - \frac{1}{3}\delta_{ij}. \tag{1}$$

is taken to be a constant:

$$\frac{db_{ij}}{dt} = \frac{\partial b_{ij}}{\partial t} + U_k \frac{\partial b_{ij}}{\partial x_k} \approx 0. \tag{2}$$

In the above equations, repeated indices denote summation, U_i and u_i represent the mean and fluctuating velocity fields and K is the turbulent kinetic energy. The weak-equilibrium condition (equation 2) is exact for homogeneous flows at equilibrium and a reasonable approximation for slowly-evolving flows. However, it should be borne in mind that the model obtained invoking this assumption will be used in inhomogeneous flows which may be far from equilibrium.

When the Reynolds stress transport equation is subject to this 'weak-equilibrium' assumption, a set of simultaneous non-linear algebraic equations is obtained. Rodi [1] proposed that this set of non-linear equations for Reynolds stresses be solved numerically. The iterative numerical solution of the set of algebraic equations can be computationally expensive, nullifying the advantage of a two-equation model over second-order closures. By presuming that the anisotropy stress tensor has the form dictated by representation theory, Pope [2] obtains semi-explicit solutions for the Reynolds stresses. In order to completely close the expression for the Reynolds stresses, Pope's methodology requires the numerical solution of a single non-linear equation for the production to dissipation ratio.

In an effort to obtain a completely explicit expression for the Reynolds stresses from the set on non-linear equations, Taulbee [3] and Gatski and Speziale [4], linearize the problem by treating the production of kinetic energy (which is the main source of the non-linearity) as a known quantity to be specified. Again, using representation theory, explicit solutions to the set of the now-linearized algebraic equations are obtained. This results in an effective viscosity that is a function of the following:

$$\nu_T = \nu_T(\frac{P}{\varepsilon}, S_{ij}, W_{ij}), \tag{3}$$

where, ε is the dissipation rate of kinetic energy, and S_{ij} and W_{ij} are the normalized strain and rotation rate of the fluid. This type of algebraic Reynolds stress model requires that P/ε be specified externally: the ratio is typically set at its equilibrium value [4]. While the linearization of the Reynolds stress equation about the equilibrium value of (P/ε) ratio is reasonable if the flow is near equilibrium, the resulting model can be internally inconsistent when used away from equilibrium. Consider the example of an arbitrary two-dimensional mean flow. Let $[P/\varepsilon]_{eq}$ be the equilibrium value of the ratio used to calculate the turbulent viscosity. The production to dissipation ratio implied by the model can then be calculated as follows:

$$[\frac{P}{\varepsilon}]_{model} = \frac{-2\overline{u_i u_j}S^*_{ij}}{\varepsilon} = \frac{2\nu_T S^*_{ij}S^*_{ij}}{\varepsilon} = 2\frac{S^*_{ij}S^*_{ij}}{\varepsilon}\nu_T([\frac{P}{\varepsilon}]_{eq}, S_{ij}, W_{ij}), \tag{4}$$

where S^*_{ij} is the dimensional strain rate. For an arbitrary flow away from equilibrium, S_{ij} and W_{ij} can be specified without restriction: *there is no guarantee that the production to dissipation ratio calculated from the algebraic Reynolds stress model will be even close to the equilibrium value assumed to calculate the turbulent viscosity.*

The current algebraic Reynolds stress models are, therefore, either self-consistent but not fully explicit (Rodi[1], Pope[2]) or explicit but not always self-consistent. The premise of this brief paper is that an algebraic Reynolds stress closure model can be of practical value if and only if the model expression is fully explicit, self-consistent and non-singular, and hence, computable in situations away from equilibrium. Towards that end, an algebraic expression for Reynolds stress which has all the above attributes is derived. It is also demonstrated that this expression is indeed the exact solution to the two-dimensional mean flow Reynolds transport equation in the weak equilibrium limit for all linear and some quasi-linear pressure strain models. This model can be very useful for the calculation of complex flows, especially in situations where the Rodi [1] algebraic equations are still being solved numerically.

2 The non-linear algebraic Reynolds stress equations

The exact Reynolds stress transport equation in an arbitrary non-inertial reference frame undergoing a rotation with angular velocity Ω_i is given by

$$\frac{\partial \overline{u_i u_j}}{\partial t} + U_k \overline{u_i u_j}_{,k} + 2\Omega_m (e_{mkj}\overline{u_i u_k} + e_{mki}\overline{u_j u_k}) + \overline{u_i u_j u_k}_{,k} = P_{ij} + \varepsilon_{ij} + \phi_{ij} + \mathcal{D}_{ij}, \tag{5}$$

where e_{ijk} is the alternating tensor. The terms, respectively, are the time rate of change, advection, Coriolis acceleration, turbulent transport, production (P_{ij}), dissipation (ε_{ij}), pressure-strain correlation (ϕ_{ij}) and pressure-viscous diffusion ($\mathcal{D}_{ij}$) of the Reynolds stress:

$$\begin{aligned} P_{ij} &= -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} \\ \varepsilon_{ij} &= 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \\ \mathcal{D}_{ij} &= \frac{\partial}{\partial x_l}[-\overline{pu_i}\delta_{jl} - \overline{pu_j}\delta_{il} + \nu\frac{\partial \overline{u_i u_j}}{\partial x_l}]. \end{aligned} \tag{6}$$

The production and dissipation rate of turbulent kinetic energy are, respectively, $P = \frac{1}{2}P_{ii}$, and $\varepsilon = \frac{1}{2}\varepsilon_{ii}$. The dissipation rate tensor can be split into its isotropic and deviatoric parts as follows:

$$\varepsilon_{ij} = \frac{2}{3}\varepsilon\delta_{ij} + d_{ij}. \tag{7}$$

The transport equation for the anisotropy tensor in non-dimensional time is derived from equation (5):

$$\begin{aligned} \frac{db_{ij}}{dt^*} + b_{ij}(\frac{P}{\varepsilon} - 1) &= -\frac{2}{3}S_{ij} - (b_{ik}S_{kj} + S_{ik}b_{kj} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}) \\ &\quad -[b_{ik}(\overline{\omega_{jk}} + 2e_{mkj}\Omega^*_m) + b_{jk}(\overline{\omega_{ik}} + 2e_{mki}\Omega^*_m)] + \frac{1}{2}\Pi^*_{ij}. \end{aligned} \tag{8}$$

In the above equation the following normalizations have been effected using the eddy turnover time:

$$\begin{aligned}
dt^* &= \frac{\varepsilon}{K} dt \\
S_{ij} &= \frac{1}{2}\frac{K}{\varepsilon}(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}) \\
\overline{\omega_{ij}} &= \frac{1}{2}\frac{K}{\varepsilon}(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}) \\
\Omega_m^* &= \frac{K}{\varepsilon}\Omega_m \\
\Pi_{ij}^* &= \frac{K}{\varepsilon}(\phi_{ij} - d_{ij}).
\end{aligned} \tag{9}$$

It is easily seen that $\frac{P}{\varepsilon} = -2b_{mn}S_{mn}$. We consider the following type of quasi-linear pressure-strain model (that includes all linear models):

$$\Pi_{ij}^* = -(C_1^0 + C_1^1\frac{P}{\varepsilon})b_{ij} + C_2 S_{ij} + C_3(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{mn}S_{mn}\delta_{ij}) + C_4(b_{ik}W_{jk}^* + b_{jk}W_{ik}^*), \tag{10}$$

where the C's are numerical constants and

$$W_{ij}^* = \overline{\omega_{ij}} + e_{mji}\Omega_m^*. \tag{11}$$

It can be shown that most of the pressure-correlation models ([5], [6], [7]) are special cases of equation (10) near weak equilibrium. Substitution of equation (10) into equation (8) and invoking the weak equilibrium condition leads to the following *non-linear* equation for the Reynolds stresses:

$$\begin{aligned}
b_{ij}[(C_1^0 - 2) - 2(C_1^1 + 2)b_{mn}S_{mn}] &= [C_2 - \frac{4}{3}]S_{ij} + [C_3 - 2](b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{lm}S_{lm}\delta_{ij}) \\
&\quad + [C_4 - 2](b_{ik}W_{jk} + b_{jk}W_{ik}).
\end{aligned} \tag{12}$$

In the above equation W_{ij} represents the total normalized vorticity given by

$$W_{ij} = \overline{\omega_{ij}} + \frac{C_4 - 4}{C_4 - 2}e_{mji}\Omega_m^*. \tag{13}$$

For the sake of clarity, define the following:

$$L_1^0 \equiv \frac{C_1^0}{2} - 1; \quad L_1^1 \equiv C_1^1 + 2; \quad L_2 \equiv \frac{C_2}{2} - \frac{2}{3}; \qquad (14)$$
$$L_3 \equiv \frac{C_3}{2} - 1; \quad L_4 \equiv \frac{C_4}{2} - 1.$$

The *non-linear* algebraic Reynolds stress equation now takes the simple form

$$b_{ij}[L_1^0 - L_1^1 b_{mn}S_{mn}] = L_2 S_{ij} + L_3(b_{ik}S_{jk} + b_{jk}S_{ik} - \frac{2}{3}b_{lm}S_{lm}\delta_{ij}) + L_4(b_{ik}W_{jk} + b_{jk}W_{ik}). \qquad (15)$$

These equations describe the fixed points of the dynamical system of equations representing the transport of the anisotropy of the Reynolds stresses (equation 8).

3 Fully-explicit solution

At this stage, the present procedure departs from those in literature (e.g., Gatski and Speziale [4]). Rather than treat the Reynolds stress term within the square brackets on the left hand side (LHS) of equation (15) implicitly as has been done in the past, this term is now retained in its explicit form. We now appeal to the representation theory for providing the most general tensorial form of the anisotropy tensor in terms of the strain and rotation rate tensors. For details of this, now routine, procedure the reader is referred to [2], [3] and [4]. For arbitrary three-dimensional mean flows the full integrity basis is composed of ten tensors. The functional form is too cumbersome to be of practical value [4]. It is customary to restrict consideration to the more tractable case of two-dimensional mean flows and use the resultant functional form of the Reynolds stress model expression for three-dimensional flows also. For two-dimensional mean flows, the general representation of the anisotropy tensor is given by (see Gatski and Speziale [4] for details):

$$b_{ij} = G_1 S_{ij} + G_2(S_{ik}W_{kj} - W_{ik}S_{kj}) + G_3(S_{ik}S_{kj} - \frac{1}{3}S_{mn}S_{mn}\delta_{ij}), \qquad (16)$$

where, $G_1 - G_3$ are unknown coefficients which are functions of the constants of the pressure-strain model and the invariants of the strain and rotation rate tensors. In incompressible flows, these invariants are

$$\begin{aligned} \eta_1 &= S_{ij}S_{ij}, \\ \eta_2 &= W_{ij}W_{ij}. \end{aligned} \tag{17}$$

The objective now is to determine the unknown coefficients by using equation (15) as the constraint.

Determination of model coefficients. For two-dimensional mean flows, using equation (16), it is easy to show that

$$b_{mn}S_{mn} = G_1\eta_1. \tag{18}$$

Substitution of equations (16) and (18) into equation (15) yields after some manipulations

$$\begin{aligned} [G_1S_{ij} + G_2(S_{ik}W_{kj} - W_{ik}S_{kj}) &+ G_3(S_{ik}S_{kj} - \frac{1}{3}S_{mn}S_{mn})](L_1^0 - \eta_1 L_1^1 G_1) \\ &= [L_2 + \frac{\eta_1}{3}L_3G_3 + 2\eta_2 L_4 G_2]S_{ij} + \\ &\quad 2L_3G_1(S_{ik}S_{kj} - \frac{1}{3}S_{mn}S_{mn}\delta_{ij}) - \\ &\quad L_4G_1(S_{ik}W_{kj} - W_{ik}S_{kj}). \end{aligned} \tag{19}$$

Comparison of the coefficients of the tensor S_{ij} on either side of equation (19) leads to the following constraint:

$$G_1[L_1^0 - \eta_1 L_1^1 G_1] = L_2 + \frac{\eta_1}{3}L_3G_3 + 2\eta_2 L_4 G_2. \tag{20}$$

The coefficients of $(S_{ik}W_{kj} - W_{ik}S_{kj})$ and $(S_{ik}S_{kj} - \frac{1}{3}S_{mn}S_{mn}\delta_{ij})$ yield:

$$G_2 = \frac{-L_4G_1}{L_0^1 - \eta_1 L_1^1 G_1}; \;\; G_3 = \frac{2L_3G_1}{L_0^1 - \eta_1 L_1^1 G_1}. \tag{21}$$

The problem is now reduced to that of determining G_1 alone. From equations (20) and (21) we get the following cubic equation for G_1:

$$G_1(L_1^0 - \eta_1 L_1^1 G_1)^2 = L_2(L_1^0 - \eta_1 L_1^1 G_1) + [\frac{2}{3}\eta_1(L_3)^2 - 2\eta_2(L_4)^2]G_1, \tag{22}$$

which can be rewritten in the following standard form:

$$(\eta_1 L_1^1)^2 G_1^3 - (2\eta_1 L_1^0 L_1^1) G_1^2 + [(L_1^0)^2 + \eta_1 L_1^1 L_2 - \frac{2}{3}\eta_1 (L_3)^2 + 2\eta_2 (L_4)^2] G_1 - L_1^0 L_2 = 0. \qquad (23)$$

It is immediately apparent that the cubic equation degenerates into a linear equation when $\eta_1 = 0$ or $L_1^1 = 0$. For these special cases G_1 is given by

$$G_1 = \frac{L_1^0 L_2}{(L_1^0)^2 + 2\eta_2 (L_4)^2}, \quad \text{when } \eta_1 = 0, \qquad (24)$$

and

$$G_1 = \frac{L_1^0 L_2}{(L_1^0)^2 + \eta_1 L_1^1 L_2 - \frac{2}{3}\eta_1 (L_3)^2 + 2\eta_2 (L_4)^2}, \quad \text{when } L_1^1 = 0. \qquad (25)$$

General solution. For the general case, the calculation of G_1 is not straightforward. This is due to fact that the cubic equation can produce multiple real roots and the choice of the appropriate solution may be difficult to make. The proper choice among the possible roots of equation (23) is necessary.

The solution to the cubic equation (23) can be calculated following the standard procedure given in most mathematical handbooks. Define the following:

$$\begin{aligned} p &\equiv -\frac{2L_1^0}{\eta_1 L_1^1}; \quad r \equiv -\frac{L_1^0 L_2}{(\eta_1 L_1^1)^2}; \\ q &\equiv \frac{1}{(\eta_1 L_1^1)^2}[(L_1^0)^2 + \eta_1 L_1^1 L_2 - \frac{2}{3}\eta_1 (L_3)^2 + 2\eta_2 (L_4)^2]; \\ a &\equiv (q - \frac{p^3}{3}); \quad b \equiv \frac{1}{27}(2p^2 - 9pq + 27r). \end{aligned} \qquad (26)$$

The discriminant of the cubic equation can now be calculated:

$$D = \frac{b^2}{4} + \frac{a^3}{27}. \qquad (27)$$

If the discriminant is positive, the cubic equation (23) has one real and two complex roots. The choice of G_1 is obvious and we pick the real root:

$$G_1 = -\frac{p}{3} + (-\frac{b}{2} + \sqrt{D})^{\frac{1}{3}} + (-\frac{b}{2} - \sqrt{D})^{\frac{1}{3}} \ \text{ for } \ D > 0. \qquad (28)$$

When the discriminant is negative equation (23) has three real roots given by

$$\begin{aligned} G_1^{(1)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\frac{\theta}{3}), \\ G_1^{(2)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\frac{\theta}{3} + \frac{2\pi}{3}), \\ G_1^{(3)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\frac{\theta}{3} + \frac{4\pi}{3}). \end{aligned} \tag{29}$$

In the above equations θ is given by

$$cos(\theta) = \frac{-b/2}{\sqrt{-a^3/27}}. \tag{30}$$

The choice of which root to pick is now less obvious. We now need a selection criterion to uniquely determine G_1.

Selection criterion. The selection of an unique value of G_1 is based on the following criterion. Consider a calculation in which the discriminant D changes sign passing through zero. *It is, then, reasonable to require that G_1 be a continuous function of D across $D = 0$.* This requirement translates into the following selection criterion:

$$\lim_{D\to 0^+} G_1 = \lim_{D\to 0^-} G_1. \tag{31}$$

When the discriminant is nearly zero, we must have

$$a < 0; \quad \text{and} \quad \frac{b^2}{4} = \frac{-a^3}{27}, \tag{32}$$

leading to

$$(\frac{|b|}{2})^{\frac{1}{3}} = \sqrt{\frac{|a|}{3}}. \tag{33}$$

We now need to consider two separate cases: when $b > 0$, and $b < 0$.

Case 1: $b > 0$. From equation (28) we have,

$$\lim_{D\to 0^+} G_1 = -\frac{p}{3} - 2[\frac{|b|}{2}]^{\frac{1}{3}} = -\frac{p}{3} - 2\sqrt{\frac{|a|}{3}}. \tag{34}$$

From equation (30) we can infer the following:

$$cos(\theta) = \frac{-|b|/2}{\sqrt{-a^3/27}} = -1, \tag{35}$$

implying that $\theta = \pi$. Substitution of this into equation (29) yields the following:

$$\begin{aligned} G_1^{(1)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\frac{\pi}{3}) = -\frac{p}{3} + \sqrt{\frac{|a|}{3}}, \\ G_1^{(2)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\pi) = -\frac{p}{3} - 2\sqrt{\frac{|a|}{3}}, \\ G_1^{(3)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\frac{5\pi}{3}) = -\frac{p}{3} + \sqrt{\frac{|a|}{3}}. \end{aligned} \tag{36}$$

Therefore, the branch of the solution that will lead to G_1 being continuous function of the discriminant is $G_1^{(2)}$.

Case 2: $b < 0$**.** For this case we have,

$$\lim_{D\to 0^+} G_1 = -\frac{p}{3} + 2[\frac{|b|}{2}]^{\frac{1}{3}} = -\frac{p}{3} + 2\sqrt{\frac{|a|}{3}}. \tag{37}$$

We get $\theta = 0$ since,

$$cos(\theta) = \frac{|b|/2}{\sqrt{-a^3/27}} = 1. \tag{38}$$

From equation (29) one obtains:

$$\begin{aligned} G_1^{(1)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(0) = -\frac{p}{3} + 2\sqrt{\frac{|a|}{3}}, \\ G_1^{(2)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\frac{2\pi}{3}) = -\frac{p}{3} - \sqrt{\frac{|a|}{3}}, \\ G_1^{(3)} &= -\frac{p}{3} + 2\sqrt{\frac{-a}{3}}cos(\frac{5\pi}{3}) = -\frac{p}{3} + \sqrt{\frac{|a|}{3}}. \end{aligned} \tag{39}$$

The branch of the solution that is now picked is $G_1^{(1)}$.

The choice of a unique value for G_1 when multiple real roots are possible is based on the requirement that G_1 be a continuous function of the discriminant D in the neighborhood of

$D = 0$. This is an important requirement, since, in the the course of computation with the model, if $D = 0$ is encountered, there should be no discontinuity in the anisotropy tensor. It can be shown that the discriminant D is positive when strain rate dominates over rotation rates ($\eta_1 \gg \eta_2$). For example, in the plane strain case, D is always negative and $D = 0$ is never encountered. In this case, the use of the above selection criterion is somewhat questionable.

Second selection criterion. For the case when D is always negative, we employ a second selection criterion. We will require that G_1 *cannot not be always positive.* A positive value of G_1 would correspond to a negative value for turbulent kinetic energy production since $P \approx -G_1 S_{ij} S_{ij}$. In a homogeneous turbulence calculation, a negative value of production (which represents energy transfer from chaotic small scales to organized large scales) at all times would constitute a violation of the second law of thermodynamics. Furthermore, if G_1 is always positive, the model will predict only positive values for the the turbulent shear stresses in the case of plane strain flow. It is well known from direct numerical simulations that the equilibrium value of turbulent shear stress in a plane strain case is negative. Therefore, if G_1 is always positive, one cannot have a viable equilibrium turbulence state. In Figure 1, we plot the behavior of the three real roots given in equation (29) over a very wide range of η_1 and η_2 values for the case $D < 0$. It is clear from the figure that $G_1^{(1)}$ and $G^{(3)}$ are always positive and $G^{(2)}$ is always negative. (It turns out that when $D < 0$, b is always positive.) Therefore, the only physically viable root is $G_1^{(2)}$.

The behavior of the multiple roots given in Figure 1 is for the Speziale *et al.* [7] pressure-strain correlation model. Similar behavior is also observed for the pressure-strain correlation models of Launder *et al.* [5] and Gibson and Launder [6] (figures not shown).

Model for G_1. The fully explicit expression for G_1 can now be summarized as follows:

$$G_1 = \begin{cases} L_1^0 L_2/[(L_1^0)^2 + 2\eta_2(L_4)^2], & \text{for } \eta_1 = 0; \\ L_1^0 L_2/[(L_1^0)^2 + \eta_1 L_1^1 L_2 - \frac{2}{3}\eta_1(L_3)^2 + 2\eta_2(L_4)^2] & \text{for } L_1 = 0; \\ -\frac{p}{3} + (-\frac{b}{2} + \sqrt{D})^{\frac{1}{3}} + (-\frac{b}{2} - \sqrt{D})^{\frac{1}{3}}, & \text{for } D > 0; \\ -\frac{p}{3} + 2\sqrt{\frac{-a}{3}} cos(\frac{\theta}{3}), & \text{for } D < 0 \text{ and } b < 0; \\ -\frac{p}{3} + 2\sqrt{\frac{-a}{3}} cos(\frac{\theta}{3} + \frac{2\pi}{3}), & \text{for } D < 0 \text{ and } b > 0 \end{cases} \tag{40}$$

The other two coefficients G_2 and G_3 can again be easily calculated from equation (21).

In figure 2, the coefficient G_1 given by equation (40) is plotted as a function of η_1 for various values of η_2. Figures 3 and 4, provide similar plots for G_2 and G_3. It is clear from the plots that these coefficients and, therefore, the effective turbulent viscosity is non-singular.

4 Model verification

In this Section, we first provide an explicit demonstration that the derived expression is indeed the exact solution to the set of non-linear algebraic equations (15). Then we compare the equilibrium anisotropy predicted by the new model to that calculated from the Reynolds averaged Navier-Stokes calculations.

4.1 Comparison with exact solution

The basic objective of most algebraic Reynolds stress modeling procedures is to find an explicit solution to the set of simultaneous non-linear equations given by equation (15). In some simple cases, the solution to the set of equations can be directly obtained without resorting to the representation theory. For homogeneous shear, one can compare the model Reynolds stresses to those calculated directly. Any deviation of the explicit model results from the direct solution even if former compares better with experimental data is undesirable, for the modeling procedure can claim no extra source of physics.

Consider the case when the mean velocity gradient field is given by

$$\frac{\partial U_i}{\partial x_j} = S^* \delta_{i1} \delta_{j2} \tag{41}$$

The set of non-linear equations for the Reynolds stress (15) simplifies to

$$\begin{aligned}
b_{12}^3 &= -\frac{2L_1^0}{SL_1^1}b_{12}^2 - \frac{b_{12}}{2(L_1^1)^2}[2\frac{(L_1^0)^2}{S^2} + \frac{L_1^1 L_2}{S}R \\
&\quad -(\frac{L_3}{3} + L_4)(L_3 - L_4) - (\frac{L_3}{3} - L_4)(L_3 + L_4)] + \frac{L_1^0 L_2}{2S(L_1^1)^2} \\
b_{11} &= b_{12}S\frac{\frac{L}{3} + L_4}{L_1^0 - b_{12}SL_1^1} \\
b_{22} &= b_{12}S\frac{\frac{L}{3} - L_4}{L_1^0 - b_{12}SL_1^1}.
\end{aligned} \tag{42}$$

In figure 5, the various components of the anisotropy tensor calculated directly from solving the above equation are compared against those computed from the explicit algebraic Reynolds stress derived in the previous section. As can be seen, the results are indistinguishable for all values of the strain rate ratio S.

4.2 Comparison of equilibrium anisotropy

Any algebraic model, at the very least should be able to calculate the equilibrium state of anisotropy of various basic homogeneous flows. For two-dimensional homogeneous flows, this comparison is more a test of the pressure-strain model rather than of the algebraic stress modeling methodology. Clearly, the model performance will depend upon the choice of pressure-strain model: we select the model of Speziale *et al* [6]. The algebraic Reynolds stress model (ARSM) is compared against the Reynolds averaged Navier-Stokes (RANS) calculations of Speziale *et al* [7], and the experimental data of Tavoularis and Corrsin [8] for the case of homogeneous shear. The results are given in Table 1. The model agrees very well with data. In fact, the present ARSM prediction is identical with the ARSM calculations of Gatski and Speziale [4]. This is not surprising, since, the two ARSM models should indeed be identical in the equilibrium limit. However, they are very different away from equilibrium, the present model being self-consistent and non-singular.

We also perform comparison between the present ARSM and RANS calculations of Speziale *et al* [7] for the case of plane-strain (two-dimensional mean flow), and axisymmetric contraction and expansion (three-dimensional mean flows): the equilibrium values are given in Table 1. As is to be expected, the ARSM and RANS values are quite close for the case of plane-strain. In this case, the difference between the two is due to the fact that the RANS calculation employs the full non-linear pressure strain model, whereas, only the quasi-linear pressure-strain model is built into the ARSM. The good agreement between the two models in the case of three-dimensional flows is more surprising. Recall that the ARSM procedure uses an integrity basis that is complete only for two-dimensional flows. In the case of axisymmetric contraction, the equilibrium values of anisotropy predicted by the two models are extremely close. For the case of axisymmetric expansion, the agreement is not as good, but quite satisfactory. (In the algebraic model calculations, the value of the parameter SK/ε is taken from that of the full Reynolds stress model calculations.)

5 Summary and Conclusions

In this brief paper, we derive the exact solution to the Reynolds stress transport equation in the weak-equilibrium limit for two-dimensional mean flows for all linear and some quasi-linear pressure-strain correlation models. This fixed point analysis of the Reynolds stress transport equation produces three roots. When the discriminant of the cubic fixed point equation is positive, two of the roots are complex and one is real. In this case, the real root clearly is the physically realistic fixed point of the Reynolds stress transport equation. When the discriminant of the fixed point equation is negative (as it happens to be for the plane strain case), all of the three roots are real. Two of the real roots lead to a negative value of turbulent viscosity and, hence, may be unphysical. The only root that leads to a positive value of turbulent viscosity also produces the equilibrium values of anisotropy that are consistent with RANS calculations for the plane strain case. We propose this expression for the Reynolds stress as a fully explicit, self-consistent algebraic model for complex flows. It is shown that the model expression captures the equilibrium values of anisotropy quite accurately in basic three dimensional flows also. Further validation and extensive testing in complex flows is currently underway.

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Figure 1: $G^{(1)}$, $G^{(2)}$, and $G_1^{(3)}$ as functions of η_1 and η_2

Figure 2: G_1 as a function of η_1 and η_2

Figure 3: G_2 as a function of η_1 and η_2

Figure 4: G_3 as a function of η_1 and η_2

Figure 5: Comparison of model calculation and exact solution for homogeneous shear flow case. Lines refer to model calculation and symbols represent exact solution. — b_{11}, - - - b_{12}, - – - b_{22}, and – – – b_{33}.